
FURTHER MATHEMATICS

9231/02

Paper 2

For examination from 2017

MARK SCHEME

Maximum Mark: 100

Specimen

This document consists of **16** printed pages.

Mark Scheme Notes

Marks are of the following three types:

- M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.
- A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).
- B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.
- The symbol \surd implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.
- Note: B2 or A2 means that the candidate can earn 2 or 0.
B2/1/0 means that the candidate can earn anything from 0 to 2.

The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.
- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking g equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

- AEF Any Equivalent Form (of answer is equally acceptable)
- AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
- BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)
- CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)
- CWO Correct Working Only – often written by a 'fortuitous' answer
- ISW Ignore Subsequent Working
- MR Misread
- PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)
- SOS See Other Solution (the candidate makes a better attempt at the same question)
- SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

- MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through $\sqrt{}$ " marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.
- PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Question	Answer	Marks	Partial Marks	Guidance
1	Find 3 independent equations for T, R_A, R_B : Resolve horizontally: $R_B = T \cos \alpha$	2	M1 A1	
	Resolve vertically: $R_A = W + T \sin \alpha$	2	M1 A1	
	Take moments about A : $R_B 3a \sin \theta = W (3a/2) \cos \theta$ $+ T a(\sin \alpha \cos \theta + \cos \alpha \sin \theta)$ or $+ T a \sin (\alpha + \theta)$ or $+ T 3a \cos \theta \sin \alpha$	2	M1 A1	(a may be omitted from moment eqns)
	Or: Take moments about B : $R_A 3a \cos \theta = W (3a/2) \cos \theta$ $+ T 2a(\sin \alpha \cos \theta + \cos \alpha \sin \theta)$ or $+ T 2a \sin (\alpha + \theta)$ or $+ T 3a \sin \theta \cos \alpha$	2	(M1 A1)	
	Or: Take moments about C : $R_A a \cos \theta + W (a/2) \cos \theta$ $= R_B 2a \sin \theta$	2	(M1 A1)	
	Or: Take moments about D : $R_A 3a \cos \theta - W (3a/2) \cos \theta$ $= R_B 3a \sin \theta$	2	(M1 A1)	
	Solve for T, R_A, R_B (AEF in W and α): $T = W / 2 \sin \alpha$ or $\frac{1}{2}W \operatorname{cosec} \alpha$ $R_A = 3W / 2$ $R_B = W / 2 \tan \alpha$ or $\frac{1}{2}W \cot \alpha$	3	B1 B1 B1	
	9			

Question	Answer	Marks	Partial Marks	Guidance
2	For A & B use conservation of momentum, e.g.: $2mv_A + mv_B = 2mu$	1	M1	(allow $2v_A + v_B = 2u$)
	Use Newton's law of restitution (consistent signs): $v_B - v_A = eu$	1	M1	
	Combine to find v_A and v_B : $v_A = (2 - e)u/3$, $v_B = 2(1 + e)u/3$	2	A1 A1	
		4		
	Find e from $v_A = v_B' $ with $v_B' = [-] 0.4 v_B$: $(2 - e) = 0.8(1 + e)$, $e = 2/3$	2	M1 A1	
	<i>EITHER</i> : Equate times in terms of reqd. distance x : $(d - x)/v_A = d/v_B + x/v_B'$ (AEF) [$v_A = v_B' = 4u/9$, $v_B = 10u/9$]	2	M1 A1	speeds need not be found:
	Use $v_A = v_B' = 0.4 v_B$ to solve for x : $d - x = 0.4d + x$, $x = 0.3d$	2	M1 A1	
	<i>OR</i> : Find dist. moved by A when B reaches wall: $d_A = (d/v_B)v_A = 0.4d$	(2)	(M1 A1)	
	Find reqd. distance x : $x = \frac{1}{2}(d - d_A) = 0.3d$	(2)	(M1 A1)	
		4		

Question	Answer	Marks	Partial Marks	Guidance
3	Find k by equating equilibrium tensions: $mg(a/2)/a = 2mg(3a/2 - ka) / ka$	2	M1 A1	(vertical motion can earn M1 only)
	$\frac{1}{2} = 3/k - 2, \quad k = 6/5 \text{ or } 1.2$	1	A1	
		3		
	Apply Newton's law at general point, e.g.: $m \frac{d^2x}{dt^2} = -mg(a/2 + x)/a$ $+ 2mg(3a/2 - ka - x) / ka$ or $m \frac{d^2y}{dt^2} = +mg(a/2 - y)/a$ $- 2mg(3a/2 - ka + y) / ka$	3	M1 A2	(lose A1 for each incorrect term)
	Simplify to give standard SHM eqn, e.g.: $\frac{d^2x}{dt^2} = - (1 + 2/k)gx / a$ $= - 8gx/3a$	1	A1	S.R.: B1 if no derivation (max 2/5)
	State or find period using $2\pi/\omega$ with $\omega = \sqrt{(8g/3a)}$: $T = 2\pi\sqrt{(3a/8g)} \text{ or } \pi\sqrt{(3a/2g)}$ or $3.85\sqrt{(a/g)} \text{ or } 1.22\sqrt{a} \text{ [s]}$	1	B1 [√]	($\sqrt{\text{ on } \omega}$)
		5		
	Substitute values in $v^2 = \omega^2(x_0^2 - x^2)$: $0.7^2 = (8g/3a)\{(0.2a)^2 - (0.05a)^2\}$	2	M1 A1	
	Solve to find numerical value of a : $0.49 = (8g/3) \times 0.0375a, \quad a = 0.49$	1	A1	
	3			

Question	Answer	Marks	Partial Marks	Guidance
4	<i>EITHER:</i> Find tension at top from $F = ma$ vertically: $T = mu^2/a - mg$	1	B1	
	<i>OR:</i> Use energy at e.g. θ to upward vertical: $\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mga(1 - \cos \theta)$ Find tension T by using $F = ma$ radially: $T = mv^2/a - mg \cos \theta$ Eliminate v^2 : $= mu^2/a + mg(2 - 3 \cos \theta)$ Find T at top by taking $\theta = 0$: $T = mu^2/a - mg$	1	(B1)	
	Find u_{\min} by requiring $T \geq 0$ at top [or $T > 0$]: $u^2/a - g \geq 0$ so $u_{\min} = \sqrt{ag}$	1	B1	A.G.
		2		
	Find v at bottom from conservation of energy: $\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mg \times 2a$ $v^2 = ag + 4ag, v = \sqrt{5ag}$	2	M1 A1	
	Find new speed V from conservation of momentum: $m'V = mv$ with $m' = m + \frac{1}{4}m$ $V = 4v/5 = 4\sqrt{ag/5}$ or $(4/5)\sqrt{5ag}$ AEF	2	M1 A1	
	4			

Question	Answer	Marks	Partial Marks	Guidance
	Find w^2 at angle θ from conservation of energy: $\frac{1}{2} m' w^2 = \frac{1}{2} m' V^2$ $- m' g a (1 + \cos \theta)$ $[w^2 = a g (6/5 - 2 \cos \theta)]$	2	M1 A1	(condone m instead of m' here since cancels out)
	S.R. Invalid energy method (max 2/5): $[\text{gives } T' = (5mg/4)(2 - 3 \cos \theta)]$ $\frac{1}{2} m' w^2 = \frac{1}{2} m u^2$ $+ m g a (1 - \cos \theta)$ $- \frac{1}{4} m g a (1 + \cos \theta)$	1	(B1)	
	Find tension T' there by using $F = ma$ radially: $T' = m' w^2 / a - m' g \cos \theta$	1	B1	
	Eliminate w^2 : $= m' V^2 / a - m' g (2 + 3 \cos \theta)$	1	A1	
	Substitute for m' and V : $= (5mg/4)(6/5 - 3 \cos \theta)$ $\text{or } 3mg/2 - (15/4) mg \cos \theta$	1	A1	AEF
		5		
	Find $\cos \theta$ when string becomes slack from $T' = 0$: $\cos \theta = \frac{1}{3} \times 6/5 = 2/5 \text{ or } 0.4$	2	M1 A1	S.R. Allow if found from $T' = mg (6/5 - 3 \cos \theta)$

Question	Answer	Marks	Partial Marks	Guidance
5	Find or use sample mean <u>and</u> estimate population variance: $x = 222.8 / 10 = 22.28$ $s^2 = 4.12 / 9$ $= 0.458 \text{ or } 103/225 \text{ or } 0.677^2$	1	M1	(allow biased here: 0.412 or 0.642 ²)
	Find confidence interval e.g.: $22.28 \pm t \sqrt{(0.458 / 10)}$	2	M1 A1	(allow z in place of t)
	Use of correct tabular value: $t_{9, 0.975} = 2.26[2]$	1	A1	
	Evaluate C.I. correct to 3 s.f.: $22.3 \pm 0.48[4] \text{ or } [21.8, 22.8]$	1	A1	
		5		
6(i)	Find prob. p of head from mean = $2 \times$ variance: $1/p = 2 \times (1-p)/p^2, \quad p = 2/3$	2	M1 A1	A.G.
6(ii)	Find $P(X=4)$ (denoting $1-p$ by $q [= 1/3]$): $P(X=4) = q^3 \times p$ $= 2/81 \text{ or } 0.0247$	1	B1	
6(iii)	Find or state $P(X > 4)$: $P(X > 4) [= 1 - (1 + q + q^2 + q^3) \times p$ $= 1 - (1 - q^4)] = q^4$ $= 1/81 \text{ or } 0.0123$	2	M1 A1	

Question	Answer	Marks	Partial Marks	Guidance
6(iv)	Formulate condition for N : $1 - q^N > 0.999$, $[(1/3)^N < 0.001]$	1	M1	
	Take logs (any base) to give bound for N : $N > \log 0.001 / \log 1/3$	1	M1	
	Find N_{\min} : $N > 6.29$, $N_{\min} = 7$	1	A1	($N < 6.29$ or $N = 6.29$ earns M2 A0)
		3		
7(i)	Find $F(x)$ for $1 \leq x \leq 4$: $F(x) = (x^3 - 1)/63$	1	B1	
	Find $G(y)$ from $Y = X^2$ for $1 \leq x \leq 4$: $G(y) = P(Y < y) = P(X^2 < y)$ $= P(X < y^{1/2}) = F(y^{1/2})$ $= (y^{3/2} - 1)/63$	2	M1 A1	(result may be stated)
	Find $g(y)$ for corresponding range of y : $g(y) = y^{1/2}/42$	1	A1	A.G.
	Find or state corresponding range of y : $1 \leq y \leq 16$	1	B1	A.G.
		5		
7(ii)	Find median value m of Y : $(m^{3/2} - 1)/63 = 1/2$ $m = 32.5^{2/3} = 10.2$	2	M1 A1	
7(iii)	Find $E(Y)$ [or equivalently $E(X^2)$]: $E(Y) = \int y g(y) dy = \int y^{3/2} dy / 42$ $= [y^{5/2}]_1^{16} / 105 = 1023/105$ $= 341/35$ or 9.74	2	M1 A1	

Question	Answer	Marks	Partial Marks	Guidance
8	Find mean of sample data [for use in Poisson distn.]: $\lambda = 220/100 = 2.2$	1	B1	
	State (at least) null hypothesis (AEF): H_0 : Poisson distn. fits data <i>or</i> $\lambda = 2.2$	1	B1	
	Find expected values $100\lambda^r e^{-\lambda}/r!$ (to 1 d.p.): 11.080 24.377 26.814 19.664 10.8151 4.759 2.491	2	M1 A1	(ignore incorrect final value here for M1)
	Combine last two cells so that exp. value ≥ 5 : O_i : 3 E_i : 7.25	1	M1*	
	Calculate value of χ^2 (to 2 d.p.; A1 dep M1*): $\chi^2 = 0.076 + 2.879 + 0.653 + 1.448$ $+ 0.441 + 2.491$ $= 7.99$	2	M1 A1	(allow 7.95 if 1 d.p. exp.values used)
	State or use consistent tabular value (to 3 s.f.): 5 cells: $\chi_{3,0.95}^2 = 7.815$ 6 cells: $\chi_{4,0.95}^2 = 9.488$ (correct) 7 cells: $\chi_{5,0.95}^2 = 11.07$	1	B1	
	State or imply valid method for conclusion e.g.: Accept H_0 if $\chi^2 <$ tabular value	1	M1	
	Conclusion (AEF, requires both values correct): Distn fits <i>or</i> $\lambda = 2.2$	1	DA1	Not combining cells [so $\chi^2 = 8.64$] can earn B1 B1 M1 A1 M0 M1 B1 M1 (max 7)
		10		

Question	Answer	Marks	Partial Marks	Guidance
9	Calculate gradient b_1 in $y - \bar{y} = b_1(x - \bar{x})$: $S_{xy} = 24\,879 - 472 \times 400/8$ $= 1\,279$ $S_{xx} = 29\,950 - 472^2/8 = 2\,102$ $b_1 = S_{xy} / S_{xx} = 0.6085$ (3 s.f.)	2	M1 A1	
	Find regression line of y on x : $y = 400/8 + b_1(x - 472/8)$ $= 50 + 0.6085(x - 59)$ $= 0.6085x + 14.1$	2	M1 A1	
	Find y when $x = 72$: $= 57.9$ or 58 Allow use of regression line of x on y (since neither variable clearly independent): $S_{yy} = 21\,226 - 400^2/8 = 1\,226$ $b_2 = S_{xy} / S_{yy} = 1.043$	2	(M1 A1)	
	$x = 472/8 + b_2(y - 400/8)$	2	(M1 A1)	
	$= 1.043y + 6.85$ $Y = 62.5$ or 62	1	A1	
		5		
	Find product moment correlation coefficient r : $r = 1\,279 / \sqrt{(2\,102 \times 1\,226)}$ or $\sqrt{(0.6085 \times 1.043)} = 0.797$	2	M1 A1*	
	State both hypotheses (B0 for $r \dots$): $H_0: \rho = 0, H_1: \rho \neq 0$	1	B1	
	State or use correct tabular two-tail r -value: $r_{8, 5\%} = 0.707$	1	B1*	
	State or imply valid method for conclusion e.g.: Reject H_0 if $ r >$ tab. value (AEF)	1	M1	

Question	Answer	Marks	Partial Marks	Guidance
	Correct conclusion : There is non-zero correlation	1	DA1	(AEF, dep A1*, B1*)
		6		

Question	Answer	Marks	Partial Marks	Guidance
10E	Find MI of lamina about Q : $I_{\text{lamina}} = \frac{1}{3}m\{(3a)^2 + (3a/2)^2\} + m(9a/2)^2$	2	M1 A1	[= (15/4 + 81/4) $ma^2 = 24 ma^2$]
	State or find MI of rod about Q : $I_{\text{rod}} = (\frac{1}{3} + 1) M (3a/2)^2$ [= $3Ma^2$]	1	B1	
	Sum to find MI of object about Q : $I_1 = 24 ma^2 + 3 Ma^2$ = $3(8m + M) a^2$	1	A1	A.G.
	Find MI of object about mid-point of PQ : $I_2 = (15/4 + 3^2) ma^2 + \frac{1}{3} M (3a/2)^2$ = $(51/4) ma^2 + \frac{3}{4} Ma^2$ = $\frac{3}{4} (17m + M) a^2$	2	M1 A1	A.G.
	Use eqn of circular motion to find $d^2\theta/dt^2$ for axis l_1 : [-] $I_1 d^2\theta/dt^2 = mg \times (9a/2) \sin \theta + Mg \times (3a/2) \sin \theta$ [= $(9m/2 + 3M/2) ga \sin \theta$]	2	M1 A1	
	[Approximate $\sin \theta$ by θ and] find ω_1^2 in SHM eqn: $\omega_1^2 = (3m + M)g / 2(8m + M) a$	1	M1	
	Find period T_1 for axis l_1 from $2\pi/\omega_1$: $T_1 = 2\pi\sqrt{2(8m + M) a / (3m + M)g}$	1	A1	(AEF)
	Use eqn of circular motion to find $d^2\theta/dt^2$ for axis l_2 : [-] $I_2 d^2\theta/dt^2 = mg \times 3a \sin \theta$	1	M1	
	[Approximate $\sin \theta$ by θ and] find ω_2^2 in SHM eqn: $\omega_2^2 = 4mg / (17m + M) a$	1	M1	
	Find period T_2 for axis l_2 from $2\pi/\omega_2$: $T_2 = 2\pi\sqrt{(17m + M) a / 4mg}$	1	A1	(AEF)
	Verify that $T_1 = T_2$ when $m = M$: (AEF) $T_1 = 2\pi\sqrt{(18 a / 4g)} = T_2$	1	B1	[Taking $m = M$ throughout 2 nd part can earn: M1 A1 M1 A0 M1 M1 A0 B1 (max 6/8)]
		8		

Question	Answer	Marks	Partial Marks	Guidance
100	State hypotheses (B0 for \bar{x} ...), e.g.: $H_0: \mu_X = \mu_Y, H_1: \mu_X \neq \mu_Y$	1	B1	
	State assumption : Distributions have equal variances	1	B1	(AEF)
	Find sample mean <u>and</u> estimate popln variances: $x = 4.2, y = 4.8$ $s_X^2 = (180 - 42^2/10) / 9$ $= 0.4 \text{ or } 0.6325^2$ $s_Y^2 = (281.5 - 57.6^2/12) / 11$ $= 0.4564 \text{ or } 251/550 \text{ or } 0.6755^2$	1	M1	(allow biased here: $0.36 \text{ or } 0.6^2$) (allow biased here: $0.4183 \text{ or } 0.6468^2$)
	Estimate (pooled) common variance: $s^2 = (9 s_X^2 + 11 s_Y^2) / 20$ <i>or</i> $(180 - 42^2/10 + 281.5 - 57.6^2/12) / 20$ $= 0.431 \text{ or } 0.6565^2$	2	M1 A1	(AEF) (note s_X^2 and s_Y^2 not needed explicitly)
	Calculate value of t (to 3 s.f.): [-] $t = (\bar{y} - \bar{x}) / s \sqrt{(1/10 + 1/12)}$ $= 2.13$	2	M1 A1	
	State or use correct tabular t value: $t_{20, 0.975} = 2.086$ [allow 2.09]	1	B1*	(or can compare $\bar{y} - \bar{x} = 0.6$ with 0.586)
	Correct conclusion: $t > 2.09$ so mean masses not same	1	DB1 [^]	(AEF, $\sqrt{\quad}$ on t , dep *B1)
	S.R. Implicitly taking s_X^2, s_Y^2 as popln. variances: $z = (\bar{y} - \bar{x}) / \sqrt{(s_X^2/10 + s_Y^2/12)}$ $= 0.6 / \sqrt{(0.078)} = 2.15$	1	(B1)	(may also earn first B1 B1 M1)
		9		

Question	Answer	Marks	Partial Marks	Guidance
	Comparison with $z_{0.975}$ and conclusion: $2.15 > 1.96$ so mean masses not same	1	(B1)	(can earn at most 5/9)
	State hypotheses (B0 for \bar{x} ...), e.g.: $H_0: \mu_X = 3.8$, $H_1: \mu_X > 3.8$ or $H_0: \mu_X = \mu_Z$, $H_1: \mu_X > \mu_Z$	1	B1	
	Calculate value of t using s_X from above: $t = (4.2 - 3.8) / (s_X / \sqrt{10}) = 2.0$	2	M1 A1	
	State or use correct tabular t value: $t_{9, 0.95} = 1.833$ [allow 1.83]	1	B1*	(or can compare 0.4 with 0.367)
	Correct conclusion: $t > 1.833$, so claim is justified or mean mass of Royals > mean mass of Crowns	1	DB1 [^]	(A.E.F., \sqrt on t , dep *B1)
		5		